Closed Form Quaternionic Least Squares Regression

Edward King Solomon

TABLE I An abbreviated list of the CODATA recommended values of the fundamental constants of physics and chemistry based on the 2014 adjustment.

Quantity	Symbol	Numerical value	Unit	Relative std. uncert. u_r
speed of light in vacuum	c, c_0	299 792 458	m s^{-1}	exact
magnetic constant	μ_0	$4\pi \times 10^{-7}$	N A^{-2}	
		$= 12.566\,370\,614... \times 10^{-7}$	N A^{-2}	exact
electric constant $1/\mu_0 c^2$	ϵ_0	$8.854\,187\,817... \times 10^{-12}$	F m^{-1}	exact
Newtonian constant of gravitation	G	$6.674\,08(31) \times 10^{-11}$	m^3 kg^{-1} s^{-2}	4.7×10^{-5}
Planck constant	h	$6.626\,070\,040(81) \times 10^{-34}$	J s	1.2×10^{-8}
$h/2\pi$	$\hbar$	$1.054\,571\,800(13) \times 10^{-34}$	J s	1.2×10^{-8}
elementary charge	e	$1.602\,176\,6208(98) \times 10^{-19}$	C	6.1×10^{-9}
magnetic flux quantum $h/2e$	Φ_0	$2.067\,833\,831(13) \times 10^{-15}$	Wb	6.1×10^{-9}
conductance quantum $2e^2/h$	G_0	$7.748\,091\,7310(18) \times 10^{-5}$	S	2.3×10^{-10}
electron mass	m_e	$9.109\,383\,56(11) \times 10^{-31}$	kg	1.2×10^{-8}
proton mass	m_p	$1.672\,621\,898(21) \times 10^{-27}$	kg	1.2×10^{-8}
proton-electron mass ratio	m_p/m_e	$1836.152\,673\,89(17)$		9.5×10^{-11}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5664(17) \times 10^{-3}$		2.3×10^{-10}
inverse fine-structure constant	α^{-1}	$137.035\,999\,139(31)$		2.3×10^{-10}
Rydberg constant $\alpha^2 m_e c/2h$	R_∞	$10\,973\,731.568\,508(65)$	m^{-1}	5.9×10^{-12}
Avogadro constant	N_A, L	$6.022\,140\,857(74) \times 10^{23}$	mol^{-1}	1.2×10^{-8}
Faraday constant $N_A e$	F	$96\,485.332\,89(59)$	C mol^{-1}	6.2×10^{-9}
molar gas constant	R	$8.314\,4598(48)$	J mol^{-1} K^{-1}	5.7×10^{-7}
Boltzmann constant R/N_A	k	$1.380\,648\,52(79) \times 10^{-23}$	J K^{-1}	5.7×10^{-7}
Stefan-Boltzmann constant $(\pi^2/60)k^4/\hbar^3c^2$	σ	$5.670\,367(13) \times 10^{-8}$	W m^{-2} K^{-4}	2.3×10^{-6}
		Non-SI units accepted for use with the SI		
electron volt (e/C) J	eV	$1.602\,176\,6208(98) \times 10^{-19}$	J	6.1×10^{-9}
(unified) atomic mass unit $\frac{1}{12}m(^{12}\mathrm{C})$	u	$1.660\,539\,040(20) \times 10^{-27}$	kg	1.2×10^{-8}

Quaternion Space-Time and Matter

Viktor Ariel*

In this work, we use the concept of quaternion time and demonstrate that it can be applied for description of four-dimensional space-time intervals. We demonstrate that the quaternion time interval together with the finite speed of light propagation allow for a simple intuitive understanding of the time interval measurement during arbitrary relative motion between a signal source and observer. We derive a quaternion form of Lorentz time dilation and show that the norm corresponds to the traditional expression of the Lorentz transformation and represents the measured value of time intervals, making the new theory inseparable from the theory of measurement. We determine that the space-time interval in the observer reference frame is given by a conjugate quaternion expression, which is essential for proper definition of the quaternion derivatives in the observer reference frame. Then, we apply quaternion differentiation to an arbitrary potential, which leads to generalized Lorentz force. The second quaternion derivative of the potential leads to expressions similar to generalized Maxwell equations. Finally, we apply the resulting formalism to electromagnetic and gravitational interactions and show that the new expressions are similar to the traditional expressions, with the exception of additional terms, related to scalar fields, that need further study and experimental verification. Therefore, the new mathematical approach based on Hamilton's quaternions may serve as a useful foundation of the unified theory of space-time and matter.

II. QUATERNION SPACE-TIME

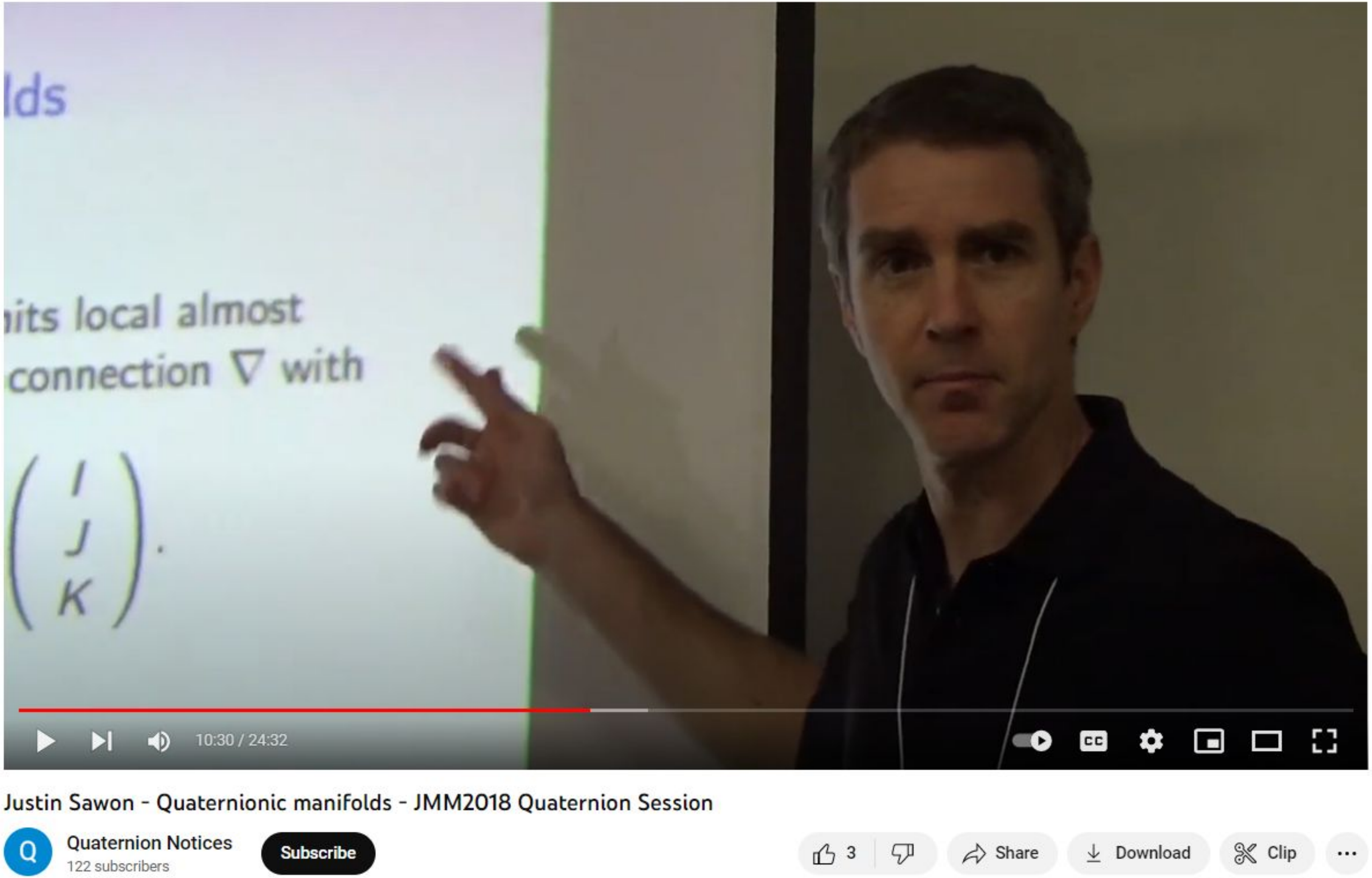
Historically, Rodrigues [1] introduced quaternions while searching for a method to describe rotation of three-dimensional solids. His discovery can be considered the precursor to quaternion algebra, which was formally introduced and extensively studied by Hamilton [2], [3], who came across quaternions while searching for mathematical division in the three-dimensional space. Hamilton was quoted: "Time is said to have only one dimension, and space to have three dimensions. The mathematical quaternion partakes of both these elements" [4]. In Hamilton's definition of quaternions, time is a real scalar and space is a three-dimensional imaginary vector, which seems like a brilliant insight predating the discovery of the four-dimensional space-time.

The key advantages of real quaternion algebra over other mathematical methods are: a positive Euclidean norm, description of both rotation and propagation in three-dimensional space, and well-defined division. Consequently, quaternion algebra deserves further investigation as an alternative mathematical formalism of space-time physics.

Since the algebra of real quaternions is the only four-dimensional division algebra, we introduce the four-dimensional quaternion manifold,

$$\boldsymbol{\tau}^4 = (\hat{\tau}_0\,, \vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3) = (\hat{\imath}_0 \tau_0\,, \vec{\imath}_1 \tau_1, \vec{\imath}_2 \tau_2, \vec{\imath}_3 \tau_3)\,, \qquad (1)$$

which we identify with time [5] in order to facilitate an intuitive physical interpretation.


lds
its local almost
connection ∇ with
I
J
K
10:30 / 24:32
Justin Sawon - Quaternionic manifolds - JMM2018 Quaternion Session
Quaternion Notices
122 subscribers
Subscribe
3
Share
Download
Clip

Finally from (11) and (12), we express the quaternion time interval in polar form,

$$\boldsymbol{t} = t\,(\cos\theta\,,\,\vec{\imath}\sin\theta) = t\,\exp\left(\vec{\imath}\theta\right), \tag{13}$$

where the angle, θ, is a function of the velocity, $\vec{v}$, and is defined as,

$$\begin{cases} \cos\theta = \dfrac{t_0}{t} = \sqrt{1-\dfrac{v^2}{c^2}}\,, \\ \\ \sin\theta = \dfrac{v}{c}. \end{cases} \tag{14}$$

Then from (13) and (14), we obtained the full polar form of the time interval transformation,

$$\begin{cases} \boldsymbol{t} = \dfrac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}} \exp\left(\vec{\imath}\theta\right) \ , \\ \\ \bar{\boldsymbol{t}} = \dfrac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}} \exp\left(-\vec{\imath}\theta\right) \ , \end{cases} \tag{15}$$

which we can a quaternion form of the Lorentz time dilation.

Les Arts Noirs: Quaternionic Least Squares Regression

Quateronic Operations via Real Matrix Identities, Quaternionic Tensor and Inverse Tensor Constructions

Edward Solomon
November 21st, 2022

Abstract

In this paper we shall arrive at a simple Closed Form Solution to Quaterionic Least Squares.

Let **C**, **A**, **B** and **Z** be quaternions, and let **C** = **AZB**, then, knowing **C**,**A**,**B** find **Z**.

The above problem is essential to solving the problem of Quaternionic Least Squares Regression, or, more generally, Hypercomplex Least Squares Regression, paraphrasing from *An Iterative Algorithm for Least Squares Problem in Quaternionic Quantum Theory*, from the *Computer Physics Communications*, **Volume 179**, **Issue 4**, **Pages 203-207**:

"*Given three quaternionic lists, the Data lists* **A** *and* **B** *and the Observation list,* **E**, *then the Quaternionic Least Squares (QLS) is the method of solving overdetermined sets of quaternion linear equations* **AXB** = **E** *that is appropriate when there is error in the matrix* **E**…*and derive an iterative method for find the minimum-norm solution of the QLS problem in quaternionic quantum theory, such that* $||AXB - \mathbf{E}|| = min.$"

Although their paper did not find a closed form solution to QLS, it uses a system similar to a Neural Network's minimization of a cost function to yield the minimum error. In this paper, we shall drive the closed form solution to QLS, and, more generally, HLS, which is Hypercomplex Least Squares, where the Hypercomplex Number has an even number of dimensions.

An interesting result of Quaternionic Least Squares is that the derivation process demonstrates that there exists not two…but three…types of Chirality: Left-handed, Right-Handed…and Middle-Handed. If there exists an undiscovered particle with Middle-Handed Chirality, then it would interact differently with matter and antimatter…such a particle would be the **Z** between **A** and **B** in the equation **C** = **AZB** , where **A** and **B** represent like measurements of a particle and its antiparticle.

The Observer, the Observed, Forward and Common Unit of Distance.

Let **O** be the **Observer**, who resides at the Origin of a Quaternionic Coordinate System, and let **P** be the **Specimen** under observation. Let the vector from **O** to **P** be the observation vector, whose direction shall be named $\vec{q}$, and its magnitude be equal to 1.

It is important that we establish the observation vector as a vector, and not the lone number 1. Multiplying a hypercomplex vector by the observation vector has no effect on the original vector, since it is an identity, in the same manner that multiplying by 1 has no effect on the original numbers.

Thus we write:

1. Forward as $+\vec{q} = \left(+\ 1\vec{q}, +\ 0\vec{i}, +\ 0\vec{j}, +\ 0\vec{k}\right)$; backwards as $-\vec{q} = \left(-\ 1\vec{q}, +\ 0\vec{i}, +\ 0\vec{j}, +\ 0\vec{k}\right)$
2. Left as $+\vec{i} = \left(+\ 0\vec{q}, +\ 1\vec{i}, +\ 0\vec{j}, +\ 0\vec{k}\right)$; right as $-\vec{i} = \left(+\ 0\vec{q}, +\ 1\vec{i}, +\ 0\vec{j}, +\ 0\vec{k}\right)$
3. Up as $+\vec{j} = \left(+\ 0\vec{q}, +\ 0\vec{i}, +\ 1\vec{j}, +\ 0\vec{k}\right)$; down as $-\vec{j} = \left(+\ 0\vec{q}, +\ 0\vec{i}, -\ 1\vec{j}, +\ 0\vec{k}\right)$
4. Counter Clockwise $+\vec{k} = \left(+\ 0\vec{q}, +\ 0\vec{i}, +\ 0\vec{j}, +\ 1\vec{k}\right)$; Clockwise as $-\vec{k} = \left(+\ 0\vec{q}, +\ 0\vec{i}, +\ 0\vec{j}, -\ 1\vec{k}\right)$

Which yields the following Cayley Table:

Quaternionic Cayley Table

Q	$\vec{q}$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{q}$	$\vec{q}$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{i}$	$-\vec{q}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$\vec{j}$	$-\vec{k}$	$-\vec{q}$	$\vec{i}$
$\vec{k}$	$\vec{k}$	$\vec{j}$	$-\vec{i}$	$-\vec{q}$

Left-Handed Quaternion vs (Improper) Right-Handed Quaternion

Clockwise vs Counterclockwise

a_1	$-a_2$	$-a_3$	$-a_4$		a_1	$-a_2$	$-a_3$	$-a_4$
a_2	a_1	a_4	$-a_3$		a_2	a_1	$-a_4$	a_3
a_3	$-a_4$	a_1	a_2		a_3	a_4	a_1	$-a_2$
a_4	a_3	$-a_2$	a_1		a_4	$-a_3$	a_2	a_1

Lemma 1.5, Right-Handed Quaternionic Division

t **A**, **B**, **C** be three sets of four real numbers, $\mathbf{A} = \{a_1,\ a_2,\ a_3,\ a_4\}$, $\mathbf{B} = \{b_1,\ b_2,\ b_3,\ b_4\}$ $\mathbf{C} = \{c_1,\ c_2,\ c_3,\ c_4\}$ then let:
$\vec{a}$ be a quaternionic vector, $\vec{a} = \left(a_1\vec{q} + a_2\vec{i} + a_3\vec{j} + a_4\vec{k}\right)$; $\vec{b}$ be a quaternionic vector, $\vec{b} = \left(b_1\vec{q} + b_2\vec{i} + b_3\vec{j} + b_4\vec{k}\right)$
$\vec{c}$ be a quaternionic vector, $\vec{c} = \left(c_1\vec{q} + c_2\vec{i} + c_3\vec{j} + c_4\vec{k}\right)$ such that, informally, $\vec{c} = \vec{a}\,\vec{b}$; properly, $\vec{c} = \vec{a}\left[\vec{b}\right]_{4R}$; then:

$\vec{a} = \vec{c}\left[\vec{w}\right]_{4R} = \vec{c}\left[\vec{b}\right]_{4R}^{-1}$, where **W** is the Involution (Inverse Matrix) of the Right-Handed **B** Matrix.

Lemma 1.10, Left-Handed Quaternionic Division

Let **A**,**B**,**C** be four sets of four real numbers, $\mathbf{A} = \{a_1, a_2, a_3, a_4\}$, $\mathbf{B} = \{b_1, b_2, b_3, b_4\}$ $\mathbf{C} = \{c_1, c_2, c_3, c_4\}$ then let:
A be a quaternionic vector, $\mathbf{A} = (a_1q + a_2i + a_3j + a_4k)$
B be a quaternionic vector, $\mathbf{B} = (b_1q + b_2i + b_3j + b_4k)$
C be a quaternionic vector, $\mathbf{C} = (c_1q + c_2i + c_3j + c_4k)$ such that $\mathbf{C} = \mathbf{AB}$; then $\mathbf{C} = [A_{4L}]\mathbf{B}$, then:
$\mathbf{B} = (V_{4L})\mathbf{C}$, where **V** is the Involution (Inverse Matrix) of the Left-Handed **A** Matrix.

Theorem 1.12, Cayley Tensor Division, C = AXB, solve for X.

Lemma 1.12a: $C = AXB$

Let **A**,**X**,**B**,**C** be four sets of four real numbers

A be a quaternionic vector, **A** $= (a_1q + a_2i + a_3j + a_4k)$

X be a quaternionic vector, **X** $= (x_1q + x_2i + x_3j + x_4k)$

B be a quaternionic vector, **B** $= (b_1q + b_2i + b_3j + b_4k)$

C be a quaternionic vector, **C** $= (c_1q + c_2i + c_3j + c_4k)$ such that **C** = **AXB**; then **X** = ($[V_{4R}][W_{4L}]$)**C**, such that:

$[V_{4R}]$ is the Involution of A_{4R}, $[V_{4R}]=[A_{4R}]^{-1}$; and that $[W_{4R}]$ is the involution of B_{4L}, $[W_{4L}]=[B_{4L}]^{-1}$.

Proof of 1.12d:

From Lemma 1.12a we have:

$$\mathbf{C} = \mathbf{AXB};\ \text{then}\ \mathbf{X} = \left(\left[V_{4R}\right]\left[W_{4L}\right] \right)\mathbf{C}$$

Then let:

$$\vec{c}_1 = \vec{a}_1 \vec{x}\, \vec{b}_1$$

$$\vec{c}_2 = \vec{a}_2 \vec{x}\, \vec{b}_2$$

Therefore: $\left(\vec{c}_1 + \vec{c}_2\right) = \left(\vec{a}_1 \vec{x}\, \vec{b}_1\right) + \left(\vec{a}_2 \vec{x}\, \vec{b}_2\right) = \left(\left[A_{1,4L}\right]\left[B_{1,4R}\right]\right)\vec{x} + \left(\left[A_{2,4L}\right]\left[B_{2,4R}\right]\right)\vec{x} = \left(\left[A_{1,4L}\right]\left[B_{1,4R}\right] + \left[A_{2,4L}\right]\left[B_{2,4R}\right]\right)\vec{x}$

Thence: $\left(\vec{c}_1 + \vec{c}_2\right) = \left(\left[A_{1,4L}\right]\left[B_{1,4R}\right] + \left[A_{2,4L}\right]\left[B_{2,4R}\right]\right)\vec{x}$, if and only if ,$\vec{x} = \left(\left[A_{1,4L}\right]\left[B_{1,4R}\right] + \left[A_{2,4L}\right]\left[B_{2,4R}\right]\right)^{-1}\left(\vec{c}_1 + \vec{c}_2\right)$

This can be extended for any number of quaternionic sums in this manner (proof by induction). This Theorem also applies to any hypercomplex numbers…thus when we solve Quaternionic Least Squares, we also solve all Hypercomplex Least Squares simultaneously.

Q.E.D

$\vec{x}_1\vec{a}_1$	$\vec{x}_2\vec{b}_1$	$\vec{x}_3\vec{c}_1$	$\vec{x}_4\vec{d}_1$	$\vec{x}_5\vec{g}_1$	$\vec{x}_6\vec{h}_1$	=	$\vec{w}_1$
$\vec{x}_1\vec{a}_2$	$\vec{x}_2\vec{b}_2$	$\vec{x}_3\vec{c}_2$	$\vec{x}_4\vec{d}_2$	$\vec{x}_5\vec{g}_2$	$\vec{x}_6\vec{h}_2$	=	$\vec{w}_2$
$\vec{x}_1\vec{a}_3$	$\vec{x}_2\vec{b}_3$	$\vec{x}_3\vec{c}_3$	$\vec{x}_4\vec{d}_3$	$\vec{x}_5\vec{g}_3$	$\vec{x}_6\vec{h}_3$	=	$\vec{w}_3$
$\vec{x}_1\vec{a}_4$	$\vec{x}_2\vec{b}_4$	$\vec{x}_3\vec{c}_4$	$\vec{x}_4\vec{d}_4$	$\vec{x}_5\vec{g}_4$	$\vec{x}_6\vec{h}_4$	=	$\vec{w}_4$
$\vec{x}_1\vec{a}_5$	$\vec{x}_2\vec{b}_5$	$\vec{x}_3\vec{c}_5$	$\vec{x}_4\vec{d}_5$	$\vec{x}_5\vec{g}_5$	$\vec{x}_6\vec{h}_5$	=	$\vec{w}_5$
$\vec{x}_1\vec{a}_6$	$\vec{x}_2\vec{b}_6$	$\vec{x}_3\vec{c}_6$	$\vec{x}_4\vec{d}_6$	$\vec{x}_5\vec{g}_6$	$\vec{x}_6\vec{h}_6$	=	$\vec{w}_6$

Theorem 3.1, Closed Form Solution to Hypercomplex Least Squares

Let n be the number of data points, let i be the index of each data point.

Let m be the number of d-hypercomplex measurements on each data point; where d is the number of hypercomplex components; let j be the index of each like measurement.

Let $\vec{x}_{i,j}$ be the j^{th} like measurement of each i^{th} data point.

Let $\mathbf{X}$ be a block matrix, with n horizontal blocks and m vertical blocks, such that each block is $d \times d$ square real number matrix, and that each (i, j) block is the form of :

1. $\left[\vec{x}_{i,j}\right]_{4L}$ if the analyst seeks the right-handed coefficient $\left[\beta_j\right]_{4R}\vec{x}_{i,j}$, or informally: $\left(x_{i,j}\right)\left(\beta_j\right)$
2. $\left[\vec{y}_{i,j,1}\right]_{4L}\left[\vec{y}_{i,j,2}\right]_{4R}$ if the analyst seeks the divine coefficient: $\left[\vec{y}_{i,j,1}\right]_{4L}\vec{\beta}_j\left[\vec{y}_{i,j,2}\right]_{4R}$, or informally: $\left(y_{i,j,1}\right)\beta_j\left(y_{i,j,2}\right)$
3. $\left[\vec{x}_{i,j}\right]_{4R}$ if the analyst seeks the left-handed coefficient $\left[\beta_j\right]_{4L}\vec{x}_{i,j}$, or informally: $\left(\beta_j\right)\left(x_{i,j}\right)$

The divine coefficient will be essential for analyzing quantum effects between particles and their respective antiparticles, when searching for some unknown mediator between them, β_j. For instance, a positron is an electron with many reversed properties, and the divine quadratic relationship of $\left(y_{i,j,1}\right)\beta_j\left(y_{i,j,2}\right)$ may be the only relationship that exists between them , where $\left(y_{i,j,1}\right)$ is some measurement of the electron and $\left(y_{i,j,2}\right)$ is some like measurement of the positron.

Let $\vec{z}_i$ be the observed response of each data point; let $\mathbf{Z}$ be a block matrix, with n horizontal blocks and only one vertical block, such that each block is $d \times 1$ real number vector, and each i^{th} block contains the vector $\vec{z}_i$.

Then we seek the optimal values of $\vec{\beta}$ such that $\mathbf{X}\vec{\beta} = \mathbf{Z}$ minimizes the collective real number magnitude of the error.

Let the set of all $\vec{x}_{i,j}$ be partitioned into three smaller sets of $\vec{a}_{i,u}$, $\vec{c}_{i,g}$, $\vec{b}_{i,v}$, where $\vec{a}_{i,u}$ contains all $\vec{x}_{i,j}$ of the left chirality, $\vec{c}_{i,g}$ contains all $\vec{x}_{i,j}$ of the divine chirality, and $\vec{b}_{i,u}$ contains all $\vec{x}_{i,j}$ of the right chirality, and let the j^{th} columns of $\mathbf{X}$ be rearranged so that all $\vec{a}_{i,u}$ precede all $\vec{c}_{i,u}$ which precede all $\vec{b}_{i,v}$ from left to right, and let this reordering be the Design Matrix $\mathbf{F}$, let μ, γ, υ be the respective lengths of those partitions, such that $m = (\mu + \gamma + \nu)$.

Now let: $[\text{ᚠ}] = \left[\mathbf{F}^{*}_{4L}\right]\left[\mathbf{F}_{4L}\right]$, where $\left[\mathbf{F}^{*}_{4L}\right]$ is the Block-Conjugate-Chiral-Transpose of $\left[\mathbf{F}_{4L}\right]$, consisting of m horizontal and vertical blocks of squared $d \times d$ real number matrices, such that each block (s, t) block contains the matrix.

1. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left[x^{*}_{m,t}\right]_{4L} \left[x_{s,t}\right]_{4L}$ if $s \leq \mu$ and $t \leq \mu$.

2. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left[x^{*}_{m,t}\right]_{4L} \left(\left[\vec{y}^{*}_{i,j,1}\right]_{4L} \left[\vec{y}^{*}_{i,j,2}\right]_{4R}\right)$ if $\mu < s \leq \gamma$. and $t \leq \mu$.

3. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left[x^{*}_{m,t}\right]_{4L} \left[x_{s,t}\right]_{4R}$ if $s > \gamma$ and $t \leq \mu$.

4. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left(\left[\vec{y}^{*}_{i,j,1}\right]_{4L} \left[\vec{y}^{*}_{i,j,2}\right]_{4R}\right) \left[x_{s,t}\right]_{4L}$ if $s \leq \mu$ and $\mu < t \leq \gamma$.

5. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left(\left[\vec{y}^{*}_{i,j,1}\right]_{4L} \left[\vec{y}^{*}_{i,j,2}\right]_{4R}\right) \left(\left[\vec{y}^{*}_{i,j,1}\right]_{4L} \left[\vec{y}^{*}_{i,j,2}\right]_{4R}\right)$ if $\mu < s \leq \gamma$ and $\mu < t \leq \gamma$.

6. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left(\left[\vec{y}^{*}_{i,j,1}\right]_{4L} \left[\vec{y}^{*}_{i,j,2}\right]_{4R}\right) \left[x_{s,t}\right]_{4R}$ if $s > \gamma$ and $\mu < t \leq \gamma$.

7. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left[x^{*}_{m,t}\right]_{4R} \left[x_{s,t}\right]_{4L}$ if $s \leq \mu$ and $t > \gamma$.

8. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left[x^{*}_{m,t}\right]_{4R} \left(\left[\vec{y}^{*}_{i,j,1}\right]_{4L} \left[\vec{y}^{*}_{i,j,2}\right]_{4R}\right)$ if $\mu < s \leq \gamma$ and $t > \gamma$.

9. $\text{ᚠ}_{s,t} = \sum_{i=1}^{i=n} \left[x^{*}_{m,t}\right]_{4R} \left[x_{s,t}\right]_{4R}$ if $s > \gamma$ and $t > \gamma$.

Let $[\text{ᛟ}] = [\text{ᚠ}]^{-1}$

Now let $[\mathbf{W}] = \left[\mathbf{F}^{*}_{4L}\right]\vec{z}$ be a block matrix, with n horizontal blocks and only one vertical block, such that each block is $d \times 1$ real number vector, and each t^{th} block contains the vector $\vec{w}_t$, such that:

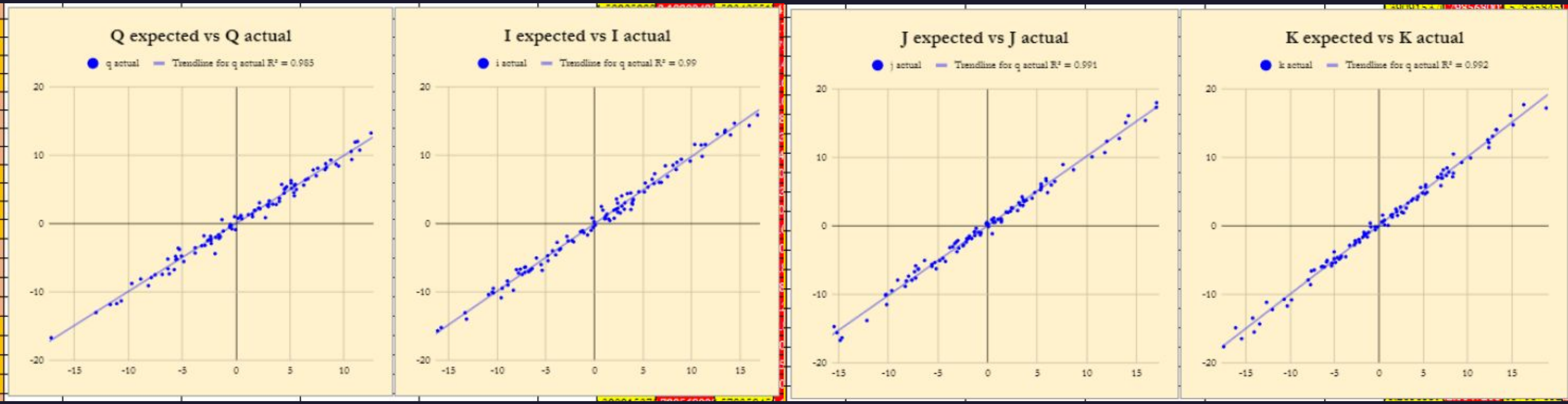
1. $\vec{w}_t = \sum_{i=1}^{i=n} \left[x^{*}_{m,t}\right]_{4L}\vec{z}$ $\qquad$ if $\qquad t \leq \mu.$
2. $\vec{w}_t = \sum_{i=1}^{i=n} \left(\left[\vec{y}^{*}_{i,j,2}\right]_{4L}\left[\vec{y}^{*}_{i,j,1}\right]_{4R}\right)\vec{z}$ $\qquad$ if $\quad \mu < t \leq \gamma$
3. $\vec{w}_t = \sum_{i=1}^{i=n} \left[x^{*}_{m,t}\right]_{4R}\vec{z}$ $\qquad$ if $\qquad t > \gamma.$

Then $\vec{\beta} = [\text{ᛟ}][\mathbf{W}]$, which is a block matrix, with n horizontal blocks and only one vertical block, such that each block is $d \times 1$ real number vector, and each t^{th} block contains the vector $\vec{\beta}_t$, which is the optimal hypercomplex coefficient for each corresponding t^{th} measurement in the data array that minimizes that collective real number magnitude of error across all n data point responses.

Q.E.D

Experiment 3.2,1, Bivariate Left-Handed Quaternionic Error

We shall first test Theorem 3.1 without error for the case of $\vec{z}_i = \left[\beta_1\right]_{4R} \vec{x}_{i,1} + \left[\beta_2\right]_{4R} \vec{x}_{i,2}$, in order to ensure that the exact simulated value of $\vec{\beta}$ is returned. After this, we shall introduce both types of error, and measure the R^2 value with several differing intensities of the introduced error.



Experiment 3.2.4, Trivariate Admixture Quaternionic Error

$$\vec{z}_i = \left(\vec{x}_{i,1}\right)\vec{\beta}_1 + \left(\vec{y}_{i,2,1}\right)\vec{\beta}_2\left(\vec{y}_{i,2,2}\right) + \vec{\beta}_3\left(\vec{x}_{i,3}\right) = \left[\vec{x}_{i,1}\right]_{4L}\vec{\beta}_1 + \left[\vec{y}_{i,2,1}\right]_{4R}\left(\left[\vec{y}_{i,2,1}\right]_{4L}\vec{\beta}_2\right) + \left[\vec{x}_{i,3}\right]_{4R}\vec{\beta}_3 \quad ; \qquad \left[\vec{y}_{i,2,1}\right]_{4R}\left(\left[\vec{y}_{i,2,1}\right]_{4L}\vec{\beta}_2\right) = \left(\left[\vec{y}_{i,2,1}\right]_{4L}\left[\vec{y}_{i,2,1}\right]_{4R}\right)\vec{\beta}_2$$

Let $\left[\mathbf{G}_{i,2}\right] = \left[\vec{y}_{i,2,1}\right]_{4L}\left[\vec{y}_{i,2,1}\right]_{4R}$ and $\left[{}^{*}\,\mathbf{G}_{i,2}\right] = \left[\vec{y}^{\,*}_{\;i,2,1}\right]_{4L}\left[\vec{y}_{\;i,2,1}\right]_{4R}$

Then we seek the following Fehu and **W** matrices, such that $\vec{\beta}$ is the product of Othala (inverse Fehu) and **W** .

$\sum_{i=1}^{i=n}\left[\vec{x}^{\,*}_{\;i,1}\right]_{4L}\left[\vec{x}_{\;i,1}\right]_{4L}$	$\sum_{i=1}^{i=n}\left[\vec{x}^{\,*}_{\;i,1}\right]_{4L}\left[\mathbf{G}_{i,2}\right]$	$\sum_{i=1}^{i=n}\left[\vec{x}^{\,*}_{\;i,1}\right]_{4L}\left[\vec{x}_{\;i,3}\right]_{4R}$	=	$\sum_{i=1}^{i=n}\left[\vec{x}^{\,*}_{\;i,1}\right]_{4L}\vec{z}_i$
$\sum_{i=1}^{i=n}\left[{}^{*}\,\mathbf{G}_{i,2}\right]\left[\vec{x}_{\;i,1}\right]_{4L}$	$\sum_{i=1}^{i=n}\left[{}^{*}\,\mathbf{G}_{i,2}\right]\left[\mathbf{G}_{i,2}\right]$	$\sum_{i=1}^{i=n}\left[{}^{*}\,\mathbf{G}_{i,2}\right]\left[\vec{x}_{\;i,3}\right]_{4R}$	=	$\sum_{i=1}^{i=n}\left[{}^{*}\,\mathbf{G}_{i,2}\right]\vec{z}_i$
$\sum_{i=1}^{i=n}\left[\vec{x}^{\,*}_{\;i,3}\right]_{4R}\left[\vec{x}_{\;i,1}\right]_{4L}$	$\sum_{i=1}^{i=n}\left[\vec{x}^{\,*}_{\;i,3}\right]_{4R}\left[\mathbf{G}_{i,2}\right]$	$\sum_{i=1}^{i=n}\left[\vec{x}^{\,*}_{\;i,3}\right]_{4R}\left[\vec{x}_{\;i,3}\right]_{4R}$	=	$\sum_{i=1}^{i=n}\left[\vec{x}^{\,*}_{\;i,3}\right]_{4R}\vec{z}_i$

Let $\vec{g}_w$ be the quaternionic output vector: $\vec{g}_w = g_{w,1}\vec{q} + g_{w,2}\vec{\imath} + g_{w,3}\vec{\jmath} + g_{w,4}\vec{k}$

Let $\vec{\alpha}_w$ be the quaternionic input vector: $\vec{\alpha}_w = \alpha_{w,1}\vec{q} + \alpha_{w,2}\vec{\imath} + \alpha_{w,3}\vec{\jmath} + \alpha_{w,4}\vec{k}$

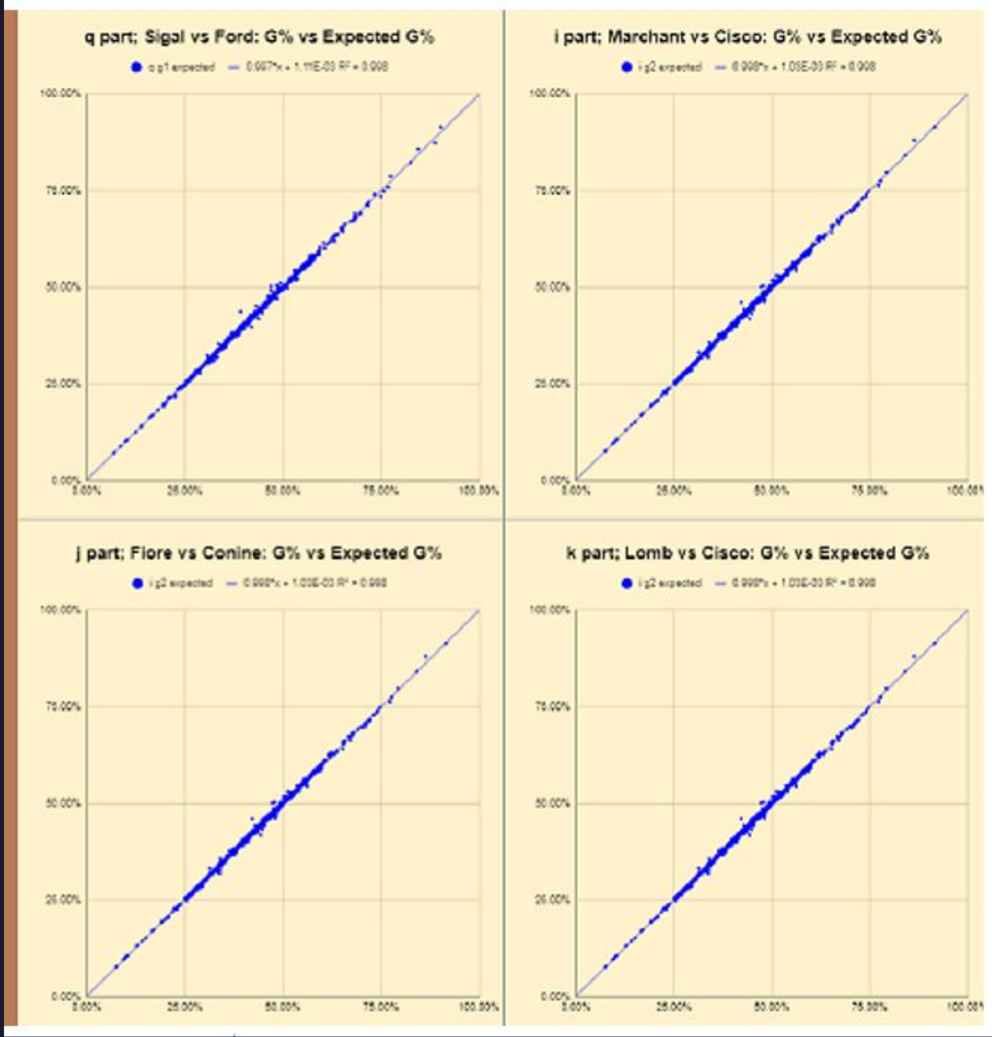
Let $\vec{h}_w$ be the quaternionic input vector: $\vec{h}_w = h_{w,1}\vec{q} + h_{w,2}\vec{\imath} + h_{w,3}\vec{\jmath} + h_{w,4}\vec{k}$

Remember that Quaternionic Multiplication is not Commutative; the following Quaternionic Manifold has an $R^2 = 0.999$

Then $\vec{g}_w = \vec{k}_{0,1} + \left(\vec{\alpha}_w\vec{k}_{1,1} + \vec{k}_{1,2}\vec{h}_w\right) + \left(\vec{k}_{2,1}(\vec{\alpha}_w^{\,2}) + (\vec{\alpha}_w\vec{h}_w)\vec{k}_{2,2} + (\vec{h}_w\vec{\alpha}_w)\vec{k}_{2,3} + \left(\vec{h}_w^{\,2}\right)\vec{k}_{2,4}\right) + \ldots$

$$\ldots + \left(\vec{k}_{3,1}(\vec{\alpha}_w^{\,3}) + (\vec{\alpha}_w^{\,2}\vec{h}_w)\vec{k}_{3,2} + (\vec{\alpha}_w)\vec{k}_{3,3}\left(\vec{h}_w\vec{\alpha}_w\right) + \left(\vec{h}_w\vec{\alpha}_w^{\,2}\right)\vec{k}_{3,4} + \left(\vec{h}_w^{\,2}\vec{\alpha}_w\right)\vec{k}_{3,5} + \left(\vec{h}_w\vec{\alpha}_w\vec{h}_w\right)\vec{k}_{3,6} + (\vec{\alpha}_w)\vec{k}_{3,7}\left(\vec{h}_w^{\,2}\right) + \vec{k}_{3,8}\left(\vec{h}_w^{\,3}\right)\right)$$

k part	$\vec{k}_{0,1}$	$\vec{k}_{1,1}$	$\vec{k}_{1,2}$	$\vec{k}_{2,1}$	$\vec{k}_{2,2}$	$\vec{k}_{2,3}$	$\vec{k}_{2,4}$	$\vec{k}_{3,1}$	$\vec{k}_{3,2}$	$\vec{k}_{3,3}$	$\vec{k}_{3,4}$	$\vec{k}_{3,5}$	$\vec{k}_{3,6}$	$\vec{k}_{3,7}$	$\vec{k}_{3,8}$
$\vec{q}$	0.0018653	1.826533	-0.8432379	0.1064711	-0.1715204	-0.197493	0.2832619	0.1685778	1.1106615	-1.5785288	-0.0191329	-0.0274986	-0.7869365	1.2282328	-0.0812626
$\vec{\imath}$	0.0016339	-0.017126	-0.0218351	0.032233	0.1086952	0.1142189	-0.2374959	0.3149882	-0.1049918	0.3109138	-0.7910358	0.66052	0.0700367	-0.2392741	-0.168964
$\vec{\jmath}$	0.002133	0.0225548	0.0086069	-0.0028455	0.2433551	0.0418391	-0.2436142	0.3155202	0.2373531	0.0266562	-1.2215577	1.5102548	-0.7232743	0.0075692	-0.0676412
$\vec{k}$	0.0028362	-0.0106878	0.0137403	-0.0567831	0.3967972	-0.3094917	-0.1366418	0.2838012	0.4149881	0.2137266	-1.4407695	1.1441195	-0.3574742	-0.2479567	-0.1042047


q part; Sigal vs Ford: G% vs Expected G%
i part; Marchant vs Cisco: G% vs Expected G%
j part; Fiore vs Conine: G% vs Expected G%
k part; Lomb vs Cisco: G% vs Expected G%

Algebraic Hexonions

6D	+q	+i	+j	+k	+t	+w
+q	+q	+i	+j	+k	+t	+w
+i	+i	-q	+k	-t	-w	+j
+j	+j	-k	-q	-w	+i	+t
+k	+k	+t	+w	-q	-j	-i
+t	+t	+w	-i	+j	-q	-k
+w	+w	-j	-t	+i	+k	-q

Algebraic Octonions

Fix 3	1	2	3	4	5	6	7	8
1	1	+2	+3	+4	+5	+6	+7	+8
2	+2	-1	+4	+5	+6	-7	-8	+3
3	+3	-4	-1	-6	+7	-8	-2	+5
4	+4	-5	+6	-1	-8	-2	-3	+7
5	+5	-6	-7	+8	-1	+3	-4	-2
6	+6	+7	+8	+2	-3	-1	-5	-4
7	+7	+8	+2	+3	+4	+5	-1	-6
8	+8	-3	-5	-7	+2	+4	+6	-1

Algebraic Sedenions

16D	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16
2	+2	-1	+4	+5	+6	-7	+8	+9	+10	-11	+12	+13	+14	-15	-16	+3
3	+3	-4	-1	-6	+7	-8	-9	-10	+11	-12	-13	-14	+15	-16	-2	+5
4	+4	-5	+6	-1	+8	+9	+10	-11	+12	+13	+14	-15	-16	-2	-3	+7
5	+5	-6	-7	-8	-1	-10	+11	-12	-13	-14	+15	-16	-2	+3	-4	+9
6	+6	+7	+8	-9	+10	-1	+12	+13	+14	-15	-16	-2	-3	-4	-5	+11
7	+7	-8	+9	-10	-11	-12	-1	-14	+15	-16	-2	+3	-4	+5	-6	+13
8	+8	-9	+10	+11	+12	-13	+14	-1	-16	-2	-3	-4	-5	-6	-7	+15
9	+9	-10	-11	-12	+13	-14	-15	+16	-1	+3	-4	+5	-6	+7	-8	-2
10	+10	+11	+12	-13	+14	+15	+16	+2	-3	-1	-5	-6	-7	-8	-9	-4
11	+11	-12	+13	-14	-15	+16	+2	+3	+4	+5	-1	+7	-8	+9	-10	-6
12	+12	-13	+14	+15	+16	+2	-3	+4	-5	+6	-7	-1	-9	-10	-11	-8
13	+13	-14	-15	+16	+2	+3	+4	+5	+6	+7	+8	+9	-1	+11	-12	-10
14	+14	+15	+16	+2	-3	+4	-5	+6	-7	+8	-9	+10	-11	-1	-13	-12
15	+15	+16	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	-1	-14
16	+16	-3	-5	-7	-9	-11	-13	-15	+2	+4	+6	+8	+10	+12	+14	-1

Algebraic Icosidgions

22D	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18	+19	+20	+21	22
2	+2	-1	+4	+5	+6	-7	+8	+9	+10	-11	+12	+13	+14	-15	+16	+17	+18	-19	+20	+21	-22	+3
3	+3	-4	-1	-6	+7	-8	-9	-10	+11	-12	-13	-14	+15	-16	-17	-18	+19	-20	-21	-22	-2	+5
4	+4	-5	+6	-1	+8	+9	+10	-11	+12	+13	+14	-15	+16	+17	+18	-19	+20	+21	-22	-2	-3	+7
5	+5	-6	-7	-8	-1	-10	+11	-12	-13	-14	+15	-16	-17	-18	+19	-20	-21	-22	-2	+3	-4	+9
6	+6	+7	+8	-9	+10	-1	+12	+13	+14	-15	+16	+17	+18	-19	+20	+21	-22	-2	-3	-4	-5	+11
7	+7	-8	+9	-10	-11	-12	-1	-14	+15	-16	-17	-18	+19	-20	-21	-22	-2	+3	-4	+5	-6	+13
8	+8	-9	+10	+11	+12	-13	+14	-1	+16	+17	+18	-19	+20	+21	-22	-2	-3	-4	-5	-6	-7	+15
9	+9	-10	-11	-12	+13	-14	-15	-16	-1	-18	+19	-20	-21	-22	-2	+3	-4	+5	-6	+7	-8	+17
10	+10	+11	+12	-13	+14	+15	+16	-17	+18	-1	+20	+21	-22	-2	-3	-4	-5	-6	-7	-8	-9	+19
11	+11	-12	+13	-14	-15	-16	+17	-18	-19	-20	-1	-22	-2	+3	-4	+5	-6	+7	-8	+9	-10	+21
12	+12	-13	+14	+15	+16	-17	+18	+19	+20	-21	+22	-1	-3	-4	-5	-6	-7	-8	-9	-10	-11	-2
13	+13	-14	-15	-16	+17	-18	-19	-20	+21	-22	+2	+3	-1	+5	-6	+7	-8	+9	-10	+11	-12	-4
14	+14	+15	+16	-17	+18	+19	+20	-21	+22	+2	-3	+4	-5	-1	-7	-8	-9	-10	-11	-12	-13	-6
15	+15	-16	+17	-18	-19	-20	+21	+22	+2	+3	+4	+5	+6	+7	-1	+9	-10	+11	-12	+13	-14	-8
16	+16	-17	+18	+19	+20	-21	+22	+2	-3	+4	-5	+6	-7	+8	-9	-1	-11	-12	-13	-14	-15	-10
17	+17	-18	-19	-20	+21	+22	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	-1	+13	-14	+15	-16	-12
18	+18	+19	+20	-21	+22	+2	-3	+4	-5	+6	-7	+8	-9	+10	-11	+12	-13	-1	-15	-16	-17	-14
19	+19	-20	+21	+22	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	-1	+17	-18	-16
20	+20	-21	+22	+2	-3	+4	-5	+6	-7	+8	-9	+10	-11	+12	-13	+14	-15	+16	-17	-1	-19	-18
21	+21	+22	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18	+19	-1	-20
22	+22	-3	-5	-7	-9	-11	-13	-15	-17	-19	-21	+2	+4	+6	+8	+10	+12	+14	+16	+18	+20	-1

Non Orientable Algebraic Quintonion Cayley Table.

H5	q	i	j	k	m
m	+j	-m	-i	-k	+q
-k	+m	+i	+k	-q	+j
-j	+i	-k	+q	-j	+m
-i	+k	+q	+j	+m	+i
-q	-q	-j	-m	-i	+k

Thank You

Edward King Solomon